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WAVELET GALERKIN SCHEME FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

ASMITA C. PATEL & V. H. PRADHAN

Department of Applied Mathematics & Humanities, Sardar Vallabhbhai National Institute of Technology, Surat, Gujarat, India

ABSTRACT

Most of the physical problems including sound waves in a viscous medium, waves in fluid filled viscous elastic tubes and magneto hydrodynamic waves in a medium with finite electrical conductivity are modeled by nonlinear partial differential equations. Many numerical and analytical methods are used to solve non-linear partial differential equations. Wavelets have generated a huge interest in different areas of applied mathematics, physics and engineering. Wavelets have been applied to the numerical solution of partial differential equations. In this paper we have developed a wavelet Galerkin scheme for non-linear partial differential equations. The present scheme is efficient, accurate and has got several advantages over other numerical methods.

KEYWORDS: Wavelet Galerkin Method, Daubechies Wavelet, Haar Wavelet, Modified Burgers' equation

INTRODUCTION

Wavelet theory [4,5] was originally applied as a powerful tool for signal and image processing. Wavelet analysis allows to obtain an efficient sparse representation of some function particularly good localization both in space and in scale of each element of a given wavelet basis. In fact, in the wavelet expansion of a function many coefficients are negligible and by discarding these coefficients we can obtain a sparse but accurate approximate representation. Moreover, using wavelet decomposition it is possible to detect singularities, irregular structure and transient phenomena exhibit by the analyzed function. In the last years, the good features of wavelets have generated a huge interest in different areas of applied mathematics, physics and engineering. Wavelets have been applied to the numerical solution of partial differential equations (PDEs). These kinds of methods have been studied both from the theoretical and the computational point of view. One of the main topics arising in the application of wavelets to the numerical algorithms for PDEs is the study of adaptivity solutions to PDEs often behave differently in different areas of their domain [4,5].

WAVELETS

A wavelet [16] is a wave-like oscillation with amplitude that starts out at zero, increases, and then decreases back to zero. It can typically be visualized as a "brief oscillation" like one might see recorded by a seismograph or heart monitor. Generally, wavelets are purposefully crafted to have specific properties that make them useful for signal processing. Wavelets can be combined, using a "shift, multiply and sum" technique called convolution, with portions of an unknown signal to extract information from the unknown signal.

TYPES OF DISCRETE WAVELET TRANSFORM

The following are the different types of discrete wavelet transform [9]

- Haar Wavelet
- Beykin
- BNC wavelets
- Coiflet
- Cohen-Daubechies-Feauvean
- Daubechies Wavelet
- Binomial-QMF
- Legendre Wavelet
- Villasenor Wavelet
- Symlet

DAUBECHIES WAVELET

Daubechies wavelets [7,9], based on the work of Ingrid Daubechies, are a family of orthogonal wavelets defining a discrete wavelet transform and characterized by a maximal number of vanishing moments for some given support. With each wavelet type of this class, there is a scaling function which generates an orthogonal multiresoution analysis.

Daubechies wavelets are chosen to have the highest number of *A* of vanishing moments for given support width *N*=*2A* and among the 2^{A-1} possible solutions one is chosen whose scaling filter has extremal phase. Daubechies wavelets are widely used in solving a broad range of a problems, e.g. self-similarity properties of a signal or fractal problems, signal discontinuities, etc. Daubechies orthogonal wavelets $D2-D20$ (even index number only) are commonly used. The index number refers to the number N of coefficients. Each wavelet has a number of zero moments or vanishing moments equal to half the number of coefficients. A vanishing moment limits the wavelet's ability to represent polynomial behavior or information in a signal.

HAAR WAVELET

In mathematics, the Haar wavelet [9] is a sequence of rescaled "square-shaped" functions which together form a wavelet family or basis. Wavelet analysis is similar to Fourier Analysis in that it allows a target function over an interval to be represented in terms of an orthonormal function basis. The Haar sequence is now recognised as the first known wavelet basis and extensively used as a teaching example.

The Haar sequence was proposed in 1909 by Alfréd Haar. Haar used these functions to give an example of an orthonormal system for the space of square-integrable functions on the unit interval [0, 1]. The study of wavelets, and even the term "wavelet", did not come until much later. As a special case of the Daubechies wavelet, the Haar wavelet is also known as **D2**. The Haar wavelet is also the simplest possible wavelet. The technical disadvantage of the Haar wavelet is that it is not continuous, and therefore not differentiable. This property can, however, be an advantage for the analysis of signals with sudden transitions, such as monitoring of tool failure in machines.

The haar wavelet's mother wavelet function $\psi(t)$ can be described as

$$\psi(t)=\begin{cases}1 & 0\le t\le 1/2,\\ -1 & 1/2\le t\le 1,\\ 0 & otherwise\end{cases}$$

Its scaling function $\phi(t)$ can be described as

$$\phi(t)=\begin{cases}1 & 0\le t<1,\\ 0 & otherwise.\end{cases}$$

WAVELET GALERKIN METHOD

The Galerkin method belongs to the family of weighted residuals methods, where the solution to a partial differential equation, $u(x,t)$ is approximated by a finite series of functions $\phi_k(x)$ as follows:

$$u(x,t)=\sum_{k=1}^{p} a_k(t)\phi_k(x) \tag{1}$$

where $\phi_k(x)$ are the basis or trial function, $a_k(t)$ are coefficients to be determined (possibly time-dependent) that satisfy the partial differential equation, and p are the number of functions. In general the approximate solution does not satisfy the partial differential equation exactly, and substituting its value results in a residual, which in turn is minimized in some sense.

Consider the following boundary value problem (BVP)[12]

$$Lu(x,t)-f(x,t)=0 \qquad ,x\in\Omega, t>0 \tag{2}$$

Substituting the trial solution (1), we get

$$L\sum_{k=1}^{p} a_k(t)\phi_k(x)-f(x,t)=R(x,t) \tag{3}$$

The method of weighted residuals minimizes $R(x,t)$ by forcing it to be zero in the domain Ω using weight functions $w_j(x)$ such that, for every weight function,

$$\int_{\Omega} R(x,t)w_j(x)\,dx=0 \qquad \text{for } j=1,2,..........,q$$

where q is the number of weight functions to be determined, depending on the boundary conditions and number of scaling functions. This discretization leads to a system of linear or non-linear ordinary differential equations where the values for $a_k(t)$ can be determined. In the Galerkin method, the weight functions are chosen to be the basis functions of Daubecheis wavelet.

In wavelet Galerkin method we choose wavelet basis as weight functions. Having multiresolutuion analysis, $V_j, j \in Z$ with scaling function $\phi(x)$, one can use $\phi_{j,k}(x)$ as the basis functions for the Galerkin method. We know that the set $\left\{\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) : k \in Z\right\}$ forms an orthonormal basis of V_j. Thus at approximation level j, one can take the orthogonal projection of u onto V_j in the following manner

$$u(x,t) \approx \Gamma_j u(x,t) = \sum_{k=1}^{p} a_{j,k}(t)\phi_{j,k}(x), \quad a_{j,k}(t) = \langle u(x,t), \phi_{j,k}(x)\rangle \tag{4}$$

and this will provide an accurate approximation to u. Furthermore, for some j, V_j will capture all details of the original function.

WAVELET GALERKIN SCHEME

PROBLEM 1: Consider the modified Burgers'equation (MBE)[13] is based upon Burgers' equation of the form

$$u_t + u^2 u_x - \upsilon u_{xx} = 0 \tag{5}$$

with initial condition

$$u(x,0) = f(x) \tag{6}$$

and boundary conditions

$$u_x(0,t) = u_x(1,t) = 0$$

$$u_{xx}(0,t) = u_{xx}(1,t) = 0 \tag{7}$$

The equation has the strong nonlinear aspects and has been used in many practical transport problems, for instance, nonlinear waves in a medium with low-frequency pumping or absorption, turbulence transport, wave processes in thermo-elastic medium, transport and dispersion of pollutants in rivers and sediment transport, and ion reflection at quasi-perpendicular shocks.

First we develop a weak formulation, from which we will derive the discretization. Multiplying both sides of Burger's equation by a test function, $\phi_{j,p} \in V$ and integrating, we get

$$\int_0^1 u_t \phi_{j,p}(x)\,dx + \int_0^1 u^2 u_x \phi_{j,p}(x)\,dx - \upsilon\int_0^1 u_{xx}\phi_{j,p}(x)\,dx = 0 \tag{8}$$

Let the trial solution of partial differential equation (8) be

$$u(x,t)=\sum_{k=0}^{L} a_k(t)\phi_{j,k}(x) \tag{9}$$

where $\phi_{j,k}(x)=2^{j/2}\phi(2^j x-k)$ and $\phi(x)$ is a scaling function of Daubechies compactly supported wavelet with vanishing moment $N/2$. Now taking test function as $\phi_{j,p}(x)=2^{j/2}\phi(2^j x-p)$, we get a system of simultaneous differential equation.

$$\int_0^1 \sum_{k=0}^{L} \dot{a}_k(t)\phi_{j,k}(x)\phi_{j,p}(x)dx+\varepsilon\int_0^1 \sum_{k=0}^{L} a_k(t)\phi_{j,k}(x)\sum_{m=0}^{L} a_m(t)\phi_{j,m}(x)\sum_{n=0}^{L} a_n(t)\phi'_{j,n}(x)\phi_{j,p}(x)dx$$
$$+\upsilon\int_0^1 \sum_{k=0}^{L} a_k(t)\phi'_{j,k}(x)\phi'_{j,p}(x)dx=0$$

$$\dot{a}_p(t)=-\sum_{k=0}^{L} a_k(t)\sum_{m=0}^{L} a_m(t)\sum_{n=0}^{L} a_n(t)\int_0^1 \phi_{j,k}(x)\phi_{j,m}(x)\phi'_{j,n}(x)\phi_{j,p}(x)dx$$
$$-\upsilon\sum_{k=0}^{L} a_k(t)\int_0^1 \phi'_{j,k}(x)\phi'_{j,p}(x)dx$$

$$\dot{a}_p(t)=-\sum_{k=0}^{L} a_k(t)\sum_{m=0}^{L} a_m(t)\sum_{n=0}^{L} a_n(t)\Omega^{0,0,1,0}_{k,m,n,p}-\upsilon\sum_{k=0}^{L} a_k(t)\Lambda^{1,1}_{k,p} \tag{10}$$

where,

$$\Omega^{0,0,1,0}_{k,m,n,p}=\int_0^1 \phi_{j,k}(x)\phi_{j,m}(x)\phi'_{j,n}(x)\phi_{j,p}(x)dx \tag{11}$$

$$\Lambda^{1,1}_{k,p}=\int_0^1 \phi'_{j,k}(x)\phi'_{j,p}(x)dx \tag{12}$$

PROBLEM 2: Consider Kortewege-de Vries-Burgers' equation [14] is a nonlinear partial differential equation, which is given by

$$u_t+\varepsilon u u_x - v u_{xx}+\mu u_{xxx}=0 \tag{13}$$

with the initial condition

$$u(x,0)=u_0(x) \tag{14}$$

and boundary conditions

$$u_x(0,t)=u_x(1,t)=0$$

$$u_{xx}(0,t)=u_{xx}(1,t)=0 \tag{15}$$

where ε, ν and μ are positive parameters. This equation was initially formulated by Gardner [2]. This model arises in many physical applications such as propagation of undular bores in shallow water waves, propagation of waves in elastic tube filled with a viscous fluid and weakly nonlinear plasma waves with certain dissipative effects. It represents long wavelength approximations where effects of the nonlinear advection term uu_x are counterbalanced by the dispersion u_{xxx}. A number of theoretical issues related the KdVB equation have received considerable attention. The traveling wave solution in particular has been studied extensively. Demiray [6], Antar and Demiray [11] derived KdVB equation as the governing evolution equation for wave propagation in fluid-filled elastic or viscoelastic tubes in which the effects of dispersion, dissipation and nonlinearity were present. Eq.(13) is combination of the Burgers' equation ($\mu = 0$) and the KdV equation ($\nu = 0$). Burgers' equation [2] was first used by Burger for the study of turbulence in 1939, whereas KdV equation [2] was first suggested by Kortewege and de Vries who used this as a nonlinear model to study the change in shape of long waves moving in a rectangular channel. KdVB equation has been solved by many authors exactly and numerically.

We develop a weak formulation, from which we will derive the discretization, multiplying both side of KdV Burgers equation by a test function, $\phi_{j,p}(x) \in V$ and integrating we

$$\int_0^1 u_t \phi_{j,p}(x)\,dx + \varepsilon \int_0^1 u u_x \phi_{j,p}(x)\,dx - \upsilon \int_0^1 u_{xx} \phi_{j,p}(x)\,dx + \mu \int_0^1 u_{xxx} \phi_{j,p}(x)\,dx = 0 \tag{16}$$

Let the trial solution of partial differential equation (13) be

$$u(x,t) = \sum_{k=0}^{L} a_k(t)\, \phi_{j,k}(x) \tag{17}$$

where $\phi_{j,k}(x) = 2^{j/2} \phi(2^j x - k)$ and $\phi(x)$ is a scaling function of Daubechies compactly supported wavelet with vanishing moment $N/2$. Now taking test function as $\phi_{j,p}(x) = 2^{j/2} \phi(2^j x - p)$, we get a system of simultaneous differential equation.

$$\int_0^1 \sum_{k=0}^{L} \dot{a}_k(t) \phi_{j,k}(x) \phi_{j,p}(x)\,dx + \varepsilon \int_0^1 \sum_{k=0}^{L} a_k(t) \phi_{j,k}(x) \sum_{m=0}^{L} a_m(t) \phi'_{j,m}(x) \phi_{j,p}(x)\,dx$$
$$+\upsilon \int_0^1 \sum_{k=0}^{L} a_k(t) \phi'_{j,k}(x) \phi'_{j,p}(x)\,dx + \mu \int_0^1 \sum_{k=0}^{L} a_k(t) \phi'_{j,k}(x) \phi''_{j,p}(x)\,dx = 0$$

$$\dot{a}_p(t) = -\varepsilon \sum_{k=0}^{L} a_k(t) \sum_{m=0}^{L} a_m(t) \int_0^1 \phi_{j,k}(x)\phi'_{j,m}(x)\phi_{j,p}(x)\,dx$$
$$-\upsilon \sum_{k=0}^{L} a_k(t) \int_0^1 \phi'_{j,k}(x)\phi'_{j,p}(x)\,dx - \mu \sum_{k=0}^{L} a_k(t) \int_0^1 \phi'_{j,k}(x)\phi''_{j,p}(x)\,dx$$

$$\dot{a}_p(t) = -\varepsilon \sum_{k=0}^{L} a_k(t) \sum_{m=0}^{L} a_m(t)\Omega^{0,1,0}_{k,m,p} - \upsilon \sum_{k=0}^{L} a_k(t)\Lambda^{1,1}_{k,p} - \mu \sum_{k=0}^{L} a_k(t)\Gamma^{1,2}_{k,p} \tag{18}$$

where,

$$\Omega^{0,1,0}_{k,m,p} = \int_0^1 \phi_{j,k}(x)\phi'_{j,m}(x)\phi_{j,p}(x)\,dx \tag{19}$$

$$\Lambda^{1,1}_{k,p} = \int_0^1 \phi'_{j,k}(x)\phi'_{j,p}(x)\,dx \tag{20}$$

$$\Gamma^{1,2}_{k,p} = \int_0^1 \phi'_{j,k}(x)\phi''_{j,p}(x)\,dx \tag{21}$$

PROBLEM 3: Consider the Newell-Whitehead- Segel equation [13] models the interaction of the effect of the diffusion term with the nonlinear effect of the reaction of the reaction term. The Newell-Whitehead- Segel equation [1] is written as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\left(1-u^2\right) \tag{22}$$

with the initial condition

$$u(x,0) = u_0(x) \tag{23}$$

and boundary conditions

$$u_x(0,t) = u_x(1,t) = 0 \tag{24}$$

Let the trial solution of the partial differential equation (22) be

$$u(x,t) = \sum_{k=0}^{L} a_k(t)\phi_{j,k}(x) \tag{25}$$

Now integrating both sides of equation (22) with respect to weight function $\phi_{j,p}(x)$ we get a system of simultaneous differential equations of first order

$$\int u_t\,\upsilon\,dx = \int u_{xx}\,\upsilon\,dx + \int u\,\upsilon\,dx - \int u\,u^2\,\upsilon\,dx$$

$$\int u_t \phi_{j,p}(x)\,dx = \int u_{xx}\,\phi_{j,p}(x)\,dx + \int u\,\phi_{j,p}(x)\,dx - \int uu^2\,\phi_{j,p}(x)\,dx$$

$$\int \sum_{k=0}^{L} a_k^{\bullet}(t)\phi_{j,k}(x)\phi_{j,p}(x)\,dx = \int \sum_{k=0}^{L} a_k(t)\phi_{j,k}^{"}(x)\phi_{j,p}(x)\,dx$$
$$+\int \sum_{k=0}^{L} a_k(t)\phi_{j,k}(x)\phi_{j,p}(x)\,dx - \int \sum_{k=0}^{L} a_k(t)a_m(t)a_s(t)\phi_{j,k}(x)\phi_{j,m}(x)\phi_{j,s}(x)\phi_{j,p}(x)\,dx$$

$$a_p^{\bullet}(t) = \sum_{k=0}^{L} a_k(t)\Lambda_{k,p}^{2,0} - \sum_{k=0}^{L} a_k(t)a_m(t)a_s(t)\Omega_{k,m,s,p}^{0,0,0,0} + a_p(t) \quad (26)$$

where

$$\Lambda_{k,p}^{2,0} = \int_{-l}^{l} \ddot{\phi}_{j,k}(x)\phi_{j,p}(x)\,dx \quad (27)$$

and

$$\Omega_{k,m,s,p}^{0,0,0,0} = \int_{-l}^{l} \phi_{j,k}(x)\phi_{j,m}(x)\phi_{j,s}(x)\phi_{j,p}(x)\,dx \quad (28)$$

CONCLUSIONS

In the present paper a new wavelet Galerkin scheme is developed for solving non-linear PDEs. Although a Wavelet Galerkin scheme is shown using Daubecheis wavelet, the same can be extended by using Haar Wavelet and coiflets Wavelets. The scheme is shown for three well-known non-linear PDEs which are arising in engineering and science. However it can be easily extended to other types of nonlinear PDEs. In the traditional finite element method the approximating functions are chosen in terms of polynomial over a finite element. With the large number of elements, the computational cost will be higher. To overcome this difficulty, the approximating functions are replaced with wavelet functions which reduce the computational cost. The present scheme is effective, efficient and can handle large number of PDEs.

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